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A new Young diagrammatic method for Kronecker products of $O(n)$ and $Sp(2m)$

Feng Pan^{†§}, Shihai Dong[‡] and J P Draayer[†]

[†] Department of Physics and Astronomy, Louisiana State University, Baton Rouge, LA 70803-4001, USA

[‡] Department of Physics, Liaoning Normal University, Dalian 116029, People's Republic of China

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Abstract. A new simple Young diagrammatic method for Kronecker products of $O(n)$ and $Sp(2m)$ is proposed based on the representation theory of Brauer algebras. A general procedure for the decomposition of the tensor products of representations for $O(n)$ and $Sp(2m)$ is outlined, which is similar to that for $U(n)$ known as the Littlewood rules, together with trace contractions from a Brauer algebra and some modification rules given by King.

1. Introduction

Representation theory of orthogonal and symplectic groups plays an important role in many areas of physics and chemistry. It arises, for example, in the description of symmetrized orbitals in quantum chemistry [1], fermion many-body theory [2], grand unification theories for elementary particles [3], supergravity [4], interacting boson and fermion dynamical symmetry models for nuclei [5–8], nuclear symplectic models [9, 10], and so on.

Reductions of Kronecker products of representations of $O(n)$ and $Sp(2m)$ groups were outlined in a series of works by King and his collaborators [11–15] based on the pioneering work of Murnaghan [16], Littlewood [17, 18], and Newell [19] on character theory and Schur functions. A similar approach was then revisited by Koike and Terada [20], in which some main points were rigorously proved. On the other hand, a Young diagrammatic method for Kronecker products for Lie groups of types B , C and D was proposed by Fischer [21]. However, as pointed out by Girardi *et al* [22, 23], rules for the decomposition of tensor products for $SO(n)$ and $Sp(2m)$ given in [21] are numerous; some of them are even ambiguous. After introducing generalized Young tableaux, with negative rows for describing $SO(2m)$, Girardi *et al* gave a formula for computing the Kronecker products for $SO(n)$ and $Sp(2m)$ in [22, 23]. The formula can be used to compute both tensor and spinor representations of $SO(n)$ and $Sp(2m)$. However, no proof was given for their formula. In [24] Littellmann proposed another Young tableau method to compute Kronecker product of some simply connected algebraic groups based on character theory. The feature of the method is that it does not use the representation theory of symmetric groups. Later, Nakashima proposed a crystal graph base [25], together with the generalized Young diagrams

[§] On leave from: Department of Physics, Liaoning Normal University, Dalian 116029, People's Republic of China

for the same problem. This method applies equally well to the q -analogue of the universal enveloping algebras of types A , B , C , and D [26].

In addition to the usefulness of these groups in many applications, the decomposition of the Kronecker products of orthogonal and symplectic groups has long been an interesting problem in mathematics, which was first considered by Weyl [27] and Brauer [28]. Besides the works mentioned above, there are many other similar ones. For example, Berele discussed a similar problem for the symplectic case in [29], and Sundaram for the orthogonal case in [30].

In this paper, we will outline a new simple Young diagrammatic method for the Kronecker products of $O(n)$ and $Sp(2m)$. Our procedure is mainly based on the induced representation of the Brauer algebra $D_f(n)$, which applies to $O(n)$ and $Sp(2m)$ because of the well known Brauer–Schur–Weyl duality relation between $D_f(n)$ and $O(n)$ or $Sp(2m)$. This relation has already enabled us to derive the Clebsch–Gordan and Racah coefficients of the quantum group $U_q(n)$ from the induction and subduction coefficients of the Hecke algebras [31, 32], and Racah coefficients of $O(n)$ and $Sp(2m)$ from the subduction coefficients of the Brauer algebra [33].

In section 2, we will give a brief introduction to Brauer algebras. Induced representations of the Brauer algebra $S_{f_1} \times S_{f_2} \uparrow D_f(n)$ will be discussed in section 3, which are important for our purposes. In section 4, we will outline a new simple Young diagrammatic method for the decomposition of the Kronecker products for $O(n)$ and $Sp(2m)$. Concluding remarks will be given in section 5.

2. Brauer algebra $D_f(n)$

The Brauer algebra $D_f(n)$ is defined algebraically by $2f - 2$ generators $\{g_1, g_2, \dots, g_{f-1}, e_1, e_2, \dots, e_{f-1}\}$ with the following relations:

$$g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1} \quad g_i g_j = g_j g_i \quad |i - j| \geq 2 \quad (1a)$$

$$e_i g_i = e_i \quad e_i g_{i-1} e_i = e_i. \quad (1b)$$

Using these defining relations and by drawing pictures of link diagrams [34, 35], one can also derive other useful ones. For example

$$e_i e_j = e_j e_i \quad |i - j| \geq 2 \quad e_i^2 = n e_i \quad (g_i - 1)^2 (g_i + 1) = 0. \quad (1c)$$

It is easy to see that $\{g_1, g_2, \dots, g_{f-1}\}$ generate a subalgebra $\mathbb{C}S_f$, which is isomorphic to the group algebra of the symmetric group; that is, $D_f(n) \supset \mathbb{C}S_f$. The properties of $D_f(n)$ have been discussed by many authors [34, 35]. Based on these results, it is known that $D_f(n)$ is semisimple, i.e. it is a direct sum of a full matrix algebra over \mathbb{C} , when n is not an integer or is an integer with $n \geq f - 1$, otherwise $D_f(n)$ is no longer semisimple. In the following, we assume that the base field is \mathbb{C} and n is an integer with $n \geq f - 1$. In this case, $D_f(n)$ is semisimple, and irreducible representations of $D_f(n)$ can be denoted by a Young diagram with $f, f - 2, f - 4, \dots, 1$ or 0 boxes. An irrep of $D_f(n)$ with $f - 2k$ boxes is denoted as $[\lambda]_{f-2k}$. The branching rule of $D_f(n) \downarrow D_{f-1}(n)$ is

$$[\lambda]_{f-2k} = \bigoplus_{[\mu] \leftrightarrow [\lambda]} [\mu] \quad (2)$$

where $[\mu]$ runs through all the diagrams obtained by removing or (if $[\lambda]$ contains less than f boxes) adding a box to $[\lambda]$. Hence, the basis vectors of $D_f(n)$ in the standard basis can

be denoted by

$$\begin{pmatrix} [\lambda]_{f-2k} & D_f(n) \\ [\mu] & D_{f-1}(n) \\ \vdots & \vdots \\ [p] & D_{f-p+1}(n) \\ [v] & D_{f-p}(n) \end{pmatrix} = \begin{pmatrix} [\lambda]_{f-2k} \\ [\mu] \\ \vdots \\ [p] \\ Y_M^{[v]} \end{pmatrix} \tag{3}$$

where $[v]$ is identical to the same irrep of S_{f-p} , $Y_M^{[v]}$ is a standard Young tableau, and M can be understood either as the Yamanouchi symbols or indices of the basis vectors in the so-called decreasing page order of the Yamanouchi symbols. Procedures for evaluating matrix elements of g_i , and e_i with $i = 1, 2, \dots, f - 1$ in the standard basis (3) have been given in [36] and [37]. It is obvious that (3) is identical to the standard basis vectors of S_f when $k = 0$. In this case, all matrix elements of e_i are zero, while the matrix elements of g_i can be obtained by the well known formula for S_f .

3. Induced representations of $D_f(n)$

From the early work of Brauer [28] and recent studies [34, 35] one knows that there is an important relation, the so-called Brauer–Schur–Weyl duality relation between the Brauer algebra $D_f(n)$ and $O(n)$ or $Sp(2m)$. If G is the orthogonal group $O(n)$ or symplectic group $Sp(2m)$, the corresponding centralizer algebra $B_f(G)$ are quotients of Brauer’s $D_f(n)$ or $D_f(-2m)$, respectively. We also need a special class of Young diagram, the so-called n -permissible Young diagram defined in [31]. A Young diagram $[\lambda]$ is said to be n -permissible if $P_\mu(n) \neq 0$ for all subdiagrams $[\mu] \leq [\lambda]$, where the subdiagrams $[\mu]$ can be obtained from $[\lambda]$ by taking away appropriate boxes, and $P_{[\mu]}(n)$ is the dimension of $O(n)$ or $Sp(2m)$ for the irrep $[\mu]$. A Young diagram $[\lambda]$ is n -permissible if and only if

- (i) its first two columns contain at most n boxes for n positive,
- (ii) it contains at most m columns for $n = -2m$ a negative even integer,
- (iii) its first two rows contain at most $2 - n$ boxes for n odd and negative.

If these conditions are satisfied, $D_f(n)$ is isomorphic to $B_f(O(n))$ for n positive, to $B_f(O(2 - n))$ for n negative and odd, and to $B(Sp(2m))$ for $n = -2m < 0$. In what follows, we assume that all irreps to be discussed are n -permissible with $n \leq f - 1$ for $n > 0$ or $-n \leq f - 1$ for negative n . These condition imply that the $D_f(n)$ being considered is semisimple.

Therefore, an irrep of $B_f(O(n))$ or $B_f(Sp(2m))$ is simultaneously the same irrep of $O(n)$ or $Sp(2m)$. However, the space of $B(G)$ and G are different. The former is labelled by its Brauer algebra indices, which operate in the $B_f(G)$ space, while the latter is labelled by its tensor components of group G . This is the so-called Brauer–Schur–Weyl duality relation between $B_f(G)$ and G , where $G = O(n)$ or $Sp(2m)$.

Hence, in order to discuss the Kronecker products of $O(n)$ and $Sp(2m)$ for the general case

$$[\lambda_1] \times [\lambda_2] \downarrow \sum_{\lambda} \{\lambda_1 \lambda_2 \lambda\} [\lambda] \tag{4}$$

where $\{\lambda_1 \lambda_2 \lambda\}$ is the number of occurrence of irrep $[\lambda]$ in the decomposition $[\lambda_1] \times [\lambda_2]$, we can consider induced representations of the Brauer algebra, $S_{f_1} \times S_{f_2} \uparrow D_f(n)$ for the same decomposition given by (4). In this case, we only need to study irreps of $D_f(n)$ induced

by irreps of $S_{f_1} \times S_{f_2}$. The standard basis vectors of $[\lambda_1]_{f_1}$, and $[\lambda_2]_{f_2}$ for S_{f_1} and S_{f_2} can be denoted by $|Y_{m_1}^{[\lambda_1]}(\omega_1^0)\rangle$, and $|Y_{m_2}^{[\lambda_2]}(\omega_2^0)\rangle$, respectively, where

$$(\omega_1^0) = (1, 2, \dots, f_1) \quad (\omega_2^0) = (f_1 + 1, f_1 + 2, \dots, f_1 + f_2) \quad (5)$$

are indices in the standard tableaux $Y_{m_1}^{[\lambda_1]}$ and $Y_{m_2}^{[\lambda_2]}$, respectively. The product of the two basis vectors are denoted by

$$|Y_{m_1}^{[\lambda_1]}, Y_{m_2}^{[\lambda_2]}, (\omega_1^0), (\omega_2^0)\rangle \equiv |Y_{m_1}^{[\lambda_1]}(\omega_1^0)\rangle |Y_{m_2}^{[\lambda_2]}(\omega_2^0)\rangle \quad (6)$$

which is called a primitive uncoupled basis vector [31, 32, 34].

The left coset decomposition of $D_f(n)$ with respect to the subalgebra $S_{f_1} \times S_{f_2}$ is denoted by

$$D_f(n) = \sum_{\omega^k} \oplus Q_{\omega}^k(S_{f_1} \times S_{f_2}) \quad (7)$$

where the left coset representatives $\{Q_{\omega}^k\}$ have two types of operations. One is the order-preserving permutations, which is the same as that for a symmetric group [31, 32]:

$$Q_{\omega}^{k=0}(\omega_1^0, \omega_2^0) = (\omega_1, \omega_2) \quad (8)$$

where

$$(\omega_1) = (a_1, a_2, \dots, a_{f_1}) \quad (\omega_2) = (a_{f_1+1}, a_{f_1+2}, \dots, a_f) \quad (9)$$

with $a_1 < a_2 < \dots < a_{f_1}$, $a_{f_1+1} < a_{f_1+2} < \dots < a_f$, and a_i represents any one of the numbers $1, 2, \dots, f$. The other, $\{Q_{\omega}^{k \geq 1}\}$, contains k -time trace contractions between two sets of indices (ω_1) and (ω_2) . For example, in $S_2 \times S_1 \uparrow D_3(n)$ for the outer product $[2] \times [1]$, there are six elements in $\{Q_{\omega}^k\}$ with

$$\{Q_{\omega}^0\} = \{1, g_2, g_1g_2\} \quad \{Q_{\omega}^1\} = \{e_2, g_1e_2, e_1g_2\}. \quad (10)$$

Let the number of operators in $\{Q_{\omega}^k\}$ be h , and the dimensions of the irreps $[\lambda_1]_{f_1} \times [\lambda_2]_{f_2}$ be $h_{[\lambda_1]}h_{[\lambda_2]}$, where $h_{[\lambda_i]}$ with $i = 1, 2$, can be computed, for example, by using Robinson's formula for the symmetric group S_f . It is obvious that the total dimension including multiple occurrences of the same irrep in the decomposition (4) is given by $hh_{[\lambda_1]}h_{[\lambda_2]}$; namely

$$hh_{[\lambda_1]}h_{[\lambda_2]} = \sum_{\lambda} \{\lambda_1\lambda_2\lambda\} \dim([\lambda]; D_f(n)) \quad (11)$$

where $\dim([\lambda]; D_f(n))$ is the dimension of $[\lambda]$ for $D_f(n)$, which was given in [29]. Hence, applying the $h Q_{\omega}^k$ to the primitive uncoupled basis vector (6), we obtain all the uncoupled basis vectors needed in the construction of the coupled basis vectors of $[\lambda]$ for $D_f(n)$, which can be denoted as

$$Q_{\omega}^k |Y_{m_1}^{[\lambda_1]}, Y_{m_2}^{[\lambda_2]}, (\omega_1^0), (\omega_2^0)\rangle = |Y_{m_1}^{[\lambda_1]}, Y_{m_2}^{[\lambda_2]}, \overbrace{(\omega_1), (\omega_2)}^k\rangle \quad (12)$$

where $\overbrace{(\omega_1), (\omega_2)}^k$ stands for k contractions between indices in (ω_1) and (ω_2) . However, all contractions between (ω_1) or (ω_2) will be zero because the $[\lambda_i]$ with $i = 1, 2$, have exactly f_i boxes, i.e. in this case, the irrep $[\lambda_i]$ of S_{f_i} is the same irrep of $D_{f_i}(n)$. Therefore, $S_{f_1} \times S_{f_2}$ can also be denoted as $D_{f_1}(n) \times D_{f_2}(n)$ when the irreps $[\lambda_i]$ for $i = 1, 2$, have exactly f_i boxes. In what follows, we will always discuss this situation, and denote $S_{f_1} \times S_{f_2}$ as $D_{f_1}(n) \times D_{f_2}(n)$ without further explanation.

Finally, the basis vectors of $[\lambda]_{f-2k}$ can be expressed in terms of the uncoupled basis vectors given by (12):

$$|[\lambda]_{f-2k}, \tau; \rho\rangle = \sum_{m_1 m_2 \omega} C_{m_1 m_2; \omega}^{[\lambda]\rho; \tau} Q_{\omega}^k |Y_{m_1}^{[\lambda_1]}(\omega_1^0), Y_{m_2}^{[\lambda_2]}(\omega_2^0)\rangle \quad (13)$$

where ρ is the multiplicity label needed in the outer-product $[\lambda_1]_{f_1} \times [\lambda_2]_{f_2} \uparrow [\lambda]_{f-2k}$, τ stands for other labels needed for the irrep $[\lambda]_{f-2k}$, and the coefficient $C_{m_1 m_2; \omega}^{[\lambda]\rho; \tau}$ is the $[\lambda_1]_{f_1} \times [\lambda_2]_{f_2} \uparrow [\lambda]_{f-2k}$ induction coefficient (IDC) or the outer-product reduction coefficient (ORC).

4. A Young diagrammatic method for the Kronecker products of $O(n)$ and $Sp(2m)$

The analytical derivation or algorithm for the IDCs discussed in section 3 is not necessary if only outer-products of $D_{f_1}(n) \times D_{f_2}(n)$ for irreps $[\lambda_1]_{f_1} \times [\lambda_2]_{f_2}$ are considered. It is obvious in (12) that irreps with $f - 2k$ boxes of $D_f(n)$ can be induced from irreps of $D_{f_1}(n) \times D_{f_2}(n)$. When $k = 0$, equation (12) is identical to that for the symmetric groups. An important operation in (12) is performed by $\{Q_{\omega}^k\}$ with $k \neq 0$. After k contractions the uncoupled primitive basis vector of $[\lambda_1]_{f_1} \times [\lambda_2]_{f_2}$ will be equivalent to the basis vectors of $[\lambda'_1]_{f_1-k} \times [\lambda'_2]_{f_2-k}$, where $[\lambda'_i]_{f_i-k}$ with $i = 1, 2$ is any possible standard Young diagram with $f_i - k$ boxes, which can be obtained from $[\lambda_i]_{f_i}$ by deleting k boxes from $[\lambda_i]$ in all possible ways. Therefore, as far as representations are concerned, the irrep $\{[\lambda]_{f-2k}\}$ of $D_f(n)$ can be obtained from the outer-product $\{[\lambda'_1]_{f_1-k} \times [\lambda'_2]_{f_2-k}\}$ of the symmetric group $S_{f_1-k} \times S_{f_2-k}$. Thus, we obtain the following rules for the outer-products of $D_{f_1}(n) \times D_{f_2}(n)$:

Lemma 1. The outer-product rule for $D_{f_1}(n) \times D_{f_2}(n) \uparrow D_f(n)$ for the decomposition

$$[\lambda_1]_{f_1} \times [\lambda_2]_{f_2} \uparrow \sum_{\lambda} \{\lambda_1 \lambda_2 \lambda\} [\lambda]$$

can be obtained diagrammatically by:

- (i) Removing k boxes, where $k = 0, 1, 2, \dots, \min(f_1, f_2)$, from $[\lambda_1]_{f_1}$ and $[\lambda_2]_{f_2}$ simultaneously in all possible ways under the following restrictions:
 - (a) Always keep the resultant diagrams $[\lambda'_i]_{f_i-k}$ with $i = 1, 2$ standard Young diagrams.
 - (b) No more than two boxes in the same column (row) in $[\lambda_1]$ with those in the same row (column) in $[\lambda_2]$ can be removed simultaneously.
- (ii) Applying the Littlewood rule for the outer-product reduction of the symmetric group to the outer-product $[\lambda'_1]_{f_1-k} \times [\lambda'_2]_{f_2-k}$, and repeatedly doing so for each k .

What we need to explain is restriction (b). Consider a simple example which is representative of the general case. Let $[\lambda_1] = [2]$, $[\lambda_2] = [1^2]$, and Q^k be a k trace contraction operator. According to our procedure, we have

$$Q^1 \left(\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} \times \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array} \right) = \left(\begin{array}{|c|c|} \hline \square & \alpha \\ \hline \end{array} \times \begin{array}{|c|} \hline \square \\ \hline \alpha \\ \hline \end{array} \right) = \left(\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array} \times \begin{array}{|c|} \hline \square \\ \hline \end{array} \right) \quad (14a)$$

while

$$Q^2 \left(\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} \times \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array} \right) = \left(\begin{array}{|c|c|} \hline \beta & \alpha \\ \hline \end{array} \times \begin{array}{|c|} \hline \beta \\ \hline \alpha \\ \hline \end{array} \right). \quad (14b)$$

The indices α and β in the boxes indicate the indices that are contracted with each other. It is known that the trace contraction of two vectors results in the symmetrization of the tensor components. Therefore, the trace contraction of anti-symmetric tensors is zero. However,

the indices of the α part is not only symmetric but also anti-symmetric with those of the β part in (14b). Hence, restriction (b) holds.

Finally, using the Brauer–Schur–Weyl duality relation between $D_f(n)$ and $O(n)$ or $Sp(2m)$, one discovers that lemma 1 also applies to the decompositions of the Kronecker products of $O(n)$ or $Sp(2m)$. Thus, we have the following lemma.

Lemma 2. The Kronecker product of $O(n)$ or $Sp(2m)$ for the decomposition given by (4) can be obtained by using procedures (i) and (ii) of lemma 1, together with the following modification rules.

For the group $O(n)$, where $n = 2l$ or $2l + 1$, ($Sp(n)$, where $n = 2l$), the resulting irrep $[\lambda] = [\lambda_1, \lambda_2, \dots, \lambda_p, \dot{0}]$ is non-standard if $p > l$. In this case, we need to remove boxes from $[\lambda]$ along a continuous boundary with hook of length $2p - n$ ($2p - n - 2$) and depth x , where x is counted by starting from the first column of $[\lambda]$ to the right-most column that the boundary hook reaches [12]. The resultant Young diagram will be admissible or set to zero if, at any stage, the removal of the required hook leaves an irregular Young diagram. Then, the resultant irrep $[\lambda]_{\text{allowed}}$ can be denoted symbolically as

$$[\lambda]_{\text{allowed}} = \begin{cases} (-)^x[\sigma] & \text{for } O(n) \\ (-)^{x+1}[\sigma] & \text{for } Sp(2m) \end{cases}$$

where $[\sigma]$ is obtained from $[\lambda]$ by using the above modification rules. For example

$$[3^3, 1] = \begin{cases} [3^3] & \text{for } O(7) \\ [3^2] & \text{for } O(4) \\ -[20] & \text{for } O(2) \\ 0 & \text{for } O(6), O(5), \text{ and } O(3) \end{cases} \quad (15)$$

which was illustrated by King [12]. In what follows, we give an example to show how this method works.

Example. Find the Kronecker product $[21] \times [11]$ for $O(n)$ or $Sp(2m)$.

First, we consider all possible diagrams with 0, 1, and $\min(f_1, f_2) = 2$ trace contractions, which are

$$\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \times \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array}, \quad \begin{array}{|c|c|} \hline \square & \times \\ \hline \square & \square \\ \hline \end{array} \times \begin{array}{|c|} \hline \square \\ \hline \times \\ \hline \end{array}, \quad \begin{array}{|c|c|} \hline \square & \square \\ \hline \times & \square \\ \hline \end{array} \times \begin{array}{|c|} \hline \square \\ \hline \times \\ \hline \end{array}, \quad \begin{array}{|c|c|} \hline \square & \times \\ \hline \times & \square \\ \hline \end{array} \times \begin{array}{|c|} \hline \times \\ \hline \times \\ \hline \end{array}. \quad (16)$$

Then, we need to compute the Kronecker products $[21] \times [11]$, $[11] \times [1]$, $[2] \times [1]$, and $[1] \times [0]$, which can be obtained by using the Littlewood rule for $U(n)$. We get

$$[21] \times [11] = [32] + [221] + [2111] + [311] \quad (17a)$$

$$[20] \times [1] = [30] + [21] \quad (17b)$$

$$[11] \times [1] = [21] + [111] \quad (17c)$$

$$[1] \times [0] = [1]. \quad (17d)$$

Finally, summing up all the irreps appearing on the right-hand sides of equations (17a)–(17d), we obtain

$$[21] \times [11] = [32] + [221] + [2111] + [311] + [30] + 2[21] + [111] + [10] \quad (18)$$

which is valid for $O(n)$ when $n \geq 8$ and $Sp(2m)$ when $m \geq 4$. Using the modification rules of lemma 2, we can easily obtain the following results

$$\begin{aligned}
 [210] \times [110] &= [320] + [221] + [211] + [311] + [300] + 2[210] \\
 &+ [111] + [100] \quad \text{for } O(7)
 \end{aligned} \tag{19a}$$

$$\begin{aligned}
 [210] \times [110] &= [320] + [221] + 3[210] + [311] + [300] + [111] \\
 &+ [100] \quad \text{for } O(6)
 \end{aligned} \tag{19b}$$

$$[21] \times [11] = [32] + [22] + [20] + [31] + [30] + 2[21] + [11] + [10] \quad \text{for } O(5) \tag{19c}$$

$$[21] \times [11] = [32] + 2[30] + 2[21] + 2[10] \quad \text{for } O(4). \tag{19d}$$

In the above computation, the following results were used:

$$[2111] = \begin{cases} [211] & \text{for } O(7) \\ [21] & \text{for } O(6) \\ [20] & \text{for } O(5) \\ [10] & \text{for } O(4) \end{cases} \tag{20a}$$

$$[221] = \begin{cases} [22] & \text{for } O(5) \\ 0 & \text{for } O(4) \end{cases} \tag{20b}$$

$$[311] = \begin{cases} [31] & \text{for } O(5) \\ [30] & \text{for } O(4) \end{cases} \tag{20c}$$

which were obtained from modification rules given in lemma 2. In addition

$$[210] \times [110] = [320] + [221] + [311] + [300] + 2[210] + [111] + [100] \quad \text{for } Sp(6) \tag{21a}$$

$$[21] \times [11] = [32] + [30] + [21] + [10] \quad \text{for } Sp(4) \tag{21b}$$

where the following modification rule were used:

$$[2111] = \begin{cases} 0 & \text{for } Sp(6) \\ -[21] & \text{for } Sp(4) \end{cases} \tag{22a}$$

$$[221] = [311] = [111] = 0 \quad \text{for } Sp(4). \tag{22b}$$

5. Concluding remarks

In this paper, a new simple Young diagrammatic method for the decomposition of the Kronecker products of $O(n)$ and $Sp(2m)$ is outlined based on the induced representation theory of $D_f(n)$. This algebra was proposed by Brauer at the end of the 1930s. His aim was indeed to solve the decomposition problem of the Kronecker products of $O(n)$ and $Sp(2m)$. On the other hand, because the representations of $D_f(n)$ are the same as those of the Birman–Wenzl algebras $C_f(r, q)$ where r and q are not roots of unity, the method also applies to the quantum groups $O_q(n)$ and $Sp_q(2m)$ where q is not a root of unity.

The induced representations of $D_f(n)$ presented in section 3 can also be used to derive the Clebsch–Gordan coefficients of $SO(n)$ when the IDCs of $D_{f_1}(n) \times D_{f_2}(n)$ are evaluated, which will be discussed in our next paper.

It should be stated that although our Young diagrammatic method for the decomposition of the $O(n)$ and $Sp(2m)$ Kronecker products is derived from the induced representation theory of Brauer algebras with the help of the Brauer–Schur–Weyl duality relation, the final results being the same as those derived by Littlewood and Newell based on character theory and Schur functions [18, 19]. In [18], the main results on how to obtain the Kronecker product of $O(n)$ and $Sp(2m)$ were achieved through the combinatorials of a certain type of S-function. However, in [18], only cases with $p \geq r$ were considered, where $n = 2p$ or $2p + 1$ for $O(n)$, and $p = m$ for $Sp(2m)$, and r is the number of rows for the corresponding irrep. In this case, no modification rule is needed, which is the same as ours. When $p \leq r$ in a Young diagram, the final diagram with a number of rows greater than p will become a non-standard irrep; the correspondence between these non-standard diagrams and the corresponding standard ones with signs in the front of the diagrams was first studied by Newell in [19], where the so-called modification rules proposed by King were given in a much simpler manner [12]. This fact is now summarized by lemma 2 of this paper.

On the other hand, the Young tableau method proposed by Littelmann [24] and crystal graph base given in [25] are related to the weight space of the corresponding Lie groups (algebras). Therefore, these methods do not use the representation theory of symmetric groups at all. However, the final results on the decomposition of the Kronecker product of $O(n)$ and $Sp(2m)$ are the same as those obtained by our Young diagrammatic method derived from Brauer algebras.

Furthermore, this method can also be applied to the Kronecker products of $SO(2l + 1)$ for any irreps and $SO(2l)$ for their irreps $[\lambda_1, \lambda_2, \dots, \lambda_k, 0]$ for $k < l$. If $k = l$, the irrep of $O(2l)$ $[\lambda_1, \lambda_2, \dots, \lambda_k]$ with $\lambda_k \neq 0$ reduces to irreps of $SO(2l)$ denoted by $[\lambda_1, \lambda_2, \dots, \lambda_k]$ and $[\lambda_1, \lambda_2, \dots, -\lambda_k]$, the dimensions of which are the same. In this case, one should be cautious and use this method. The dimension formula for $SO(n)$ is always helpful in checking the final results.

Finally, it should be noted that the method applies only to tensor or ‘true’ representations of $O(n)$. The spinor representations of $O(n)$ are related to spinor representations of Brauer algebras according to the Brauer–Schur–Weyl duality relation, which still need to be studied further.

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